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THE INFLUENCE OF MACH NUMBER AND HEAT TRANSFER ON THE TURBULENT FLOW WALL LAW

Julius Rotta

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16. Abstract

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THE INFLUENCE OF MACH NUMBER AND HEAT TRANSFER ON THE TURBULENT FLOW WALL LAW*

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Julius Rotta

SUMMARY

The similarity law of turbulent flow along a wall is extended to the case of compressible flow with heat transfer. After formulating the problem mathematically, some calculations are carried out with simplifying assumptions and compared with published experimental results. The results are discussed. Complete equations are given for the velocity and temperature distributions in the fully turbulent region.

1. Introduction

In the case of turbulent boundary layers over flat or slightly curved walls, for the case of incompressible flow, we have the variable parameters such as the pressure distribution, the wall roughness as well as the density and viscosity of the flow. Since it is impossible to experimentally analyze all of the possible combinations of these influencing parameters and to collect them into a catalogue, attempts have been made to develop calculations methods in order to cover these influences. However, turbulent flows cannot be exactly calculated. Therefore, they have to be attacked using approximate methods. A completely acceptable solution has not yet been found. For the compressible flow case,

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there are two additional influencing parameters, the temperature outside of the boundary layer and the temperature distribution at the wall. Therefore, additional material parameters of the flow medium become important. Because of the increase in number of independent variables, the requirements for calculation methods are even more important. At the same time, the problem becomes /265
more difficult. For turbulent boundary layers in the case of supersonic flow, a large number of theories have been suggested, and the results deviate sometimes greatly. This is shown by a comparison of D. R. Chapman and R. H. Kester [1] [1)]. Just as in the incompressible case, the layers very close to the wall greatly influence the magnitude of the friction resistance and the heat transfer. Therefore, in order to refine existing theories it is important to refine our knowledge about the behavior of the flow near the wall.

For turbulent flows of incompressible media, we have a universal velocity law in the regions near the wall. In the thin underlayer at the wall, where the viscosity forces and the turbulent apparent stresses are still substantial, a very large part of the turbulent fluctuation energy is produced and dissipated. Therefore, the flow conditions are not influenced by those of regions located further away from the wall. The flow velocity $\bar{u}$ averaged over time parallel to the wall, therefore, depends only on the shear stress τ_w at the wall for smooth surfaces, as well as the density ρ and the kinematic viscosity $\nu = \mu/\rho$ flow medium. It also depends on the distance to the wall y. L. Prandtl [2] in 1932 found the following similarity law after introducing the shear stress velocity $u_\tau = \sqrt{\tau_w/\rho}$

(1) $\bar{u} = u_\tau f(\eta)$

where $f(\eta)$ is a universal function of the dimensionless wall

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1) In reference 1 there is an additional list of publications up to 1953.

distance $y = y u_\tau / \nu$. This relationship was experimentally confirmed in pipes and also for boundary layers with and without a pressure gradient [3].

At high speeds and large heat transfer, this law is influenced due to the fact that because of the changing temperature $\bar{T}$, the material of values no longer remain constant. Available measurements [4] on turbulent boundary layers at hypersonic speed show, that these affects can become substantial. In the known approximation methods for treating turbulent boundary layers for subsonic flow (for example, [5,6]), this fact was not taken into account much. Even for small Mach numbers. in pipes [8] or over walls, the heat transfer is so great that the material values of the flow medium are influenced by this. In the present paper we will theoretically investigate these influences and compare calculations and measurements.

2. Similarity analysis

If we expand the Prandtl similarity law, then one has to introduce additional parameters such as the heat flux density q_w at the wall which is the amount of heat given off at the wall per unit of area and unit of time as well as the wall temperature T_w (or the related speed of sound $a = \sqrt{(\gamma - 1) k_g T_w}$. The heat transfer coefficient λ and the specific heats have to be added as material quantities. The quantities with dimensions can be summarized for the following dimensionless parameters for gases:
Mach number[2]

(2a) $M_\tau = u_\tau / a_w$

Heat transfer coefficient

(2b) $St = q_w / \varrho_w c_{pw} T_w u_\tau$

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2) After substituting the quantities for u_τ and $c_w = \sqrt{\tau_w / \varrho_w}$, it is possible to express M_τ by the square root of the ratio of the wall shear stress τ_w and pressure p_w at the wall: $M_\tau = \sqrt{\tau_w / p_w}$

Prandtl coefficient

(2c) $Pr_\infty = c_{pw}\mu_w/\lambda_w$

Ratio of specific heat

(2d) $\gamma_\infty = c_{pw}/c_{vw}$

Here we have $u_\tau = \sqrt{\tau_w/\rho_w}$ and the dimensionless wall distance is

$y_+ = y u_\tau/\nu_w$

If we are dealing with the flow of an ideal gas, and if we can assume that the viscosity μ, the heat transfer coefficient λ and the specific heats c_p, c_v are constants, although additional influences are considered, by expanding the similarity law (1) with the four parameters (2a) to (2d). However, in general the viscosity changes substantially with temperature, whereas Pr, γ and c_p first can be assumed to be constant. In this case the law $\mu(T)$ has to be included in the analysis. Using the simple power law

(3) $\mu = \mu_\infty \left(\frac{T}{T_\infty}\right)^\omega$

we have an additional parameter, the exponent w.

Under these conditions the similarity law for velocity and temperature distribution can be written as follows

(4a) $\bar{u} = u_\infty f(\eta_+, M_\infty, \beta_\infty, Pr_\infty, \gamma_\infty, \omega)$,

(4b) $\bar{T} = T_\infty g(\eta_+, M_\infty, \beta_\infty, Pr_\infty, \gamma_\infty, \omega)$.

Here $\bar{T}$ is the average of the absolute temperature[3]. The problem becomes much more complicated if very high temperatures occur. Then the molecular structure of the gas changes because of dissociation or ionization. However, we will not discuss that here. Also, we will restrict ourselves to the case of a smooth surface. If the wall consists of a material with a poor heat conductivity, then the fluctuations of the temperature gradient at the wall can produce fluctuations in the wall temperature, which then can have a feedback effect on the flow due to the material values of the flow medium. This influence, however, is

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3) In the case of changing specific heat, it would be better to use the enthalpy $h = \int c_p dT$

excluded by assuming a wall with large heat conductivity. For the investigated measurement results this condition is satisfied. The fluctuation in wall temperature can then be ignored.

The range of validity of the wall law is specified by the fact that according to assumption, all derivatives of average values in the main flow direction x (for example, $\partial u/\partial x$, $\partial p/\partial x$) do not have any noticeable influence on the flow within the layer in question. This is only true for boundary layers inside a layer thickness which is small compared with the total thickness of the boundary layer ($y \ll \delta$); . For flows in pipes this is true for layer thicknesses which are small compared with the size of the pipe (pipe radius). For turbulent boundary layers of incompressible flow according to [3] the velocity distribution found in experiments follows the universal wall law in a layer with a thickness of about 1/5 of the boundary layer thickness. One well known fact is that for supersonic boundary layers, the lower layer influenced by the viscosity is relatively thick compared with the incompressible case. This is related to the temperature increase at or in the vicinity of the wall, which then results in a corresponding increase in the kinematic viscosity. For this reason, the assumptions for the validity of the wall law at high Mach numbers in general are less well satisfied than for incompressible boundary layers with the same Reynolds numbers (refer to the outer flow). This is especially true for the published test results. However, everything depends on the prevailing Reynolds number, and one cannot generalize and say that the assumptions for the wall law are always poorly satisfied at high Mach numbers. /266

We are not interested in the influence of the two changing parameters M_τ and β_{qi}. The other parameters only represent material values. which are constant under the test conditions. The parameters M_τ and β_q are used in a similar manner as in investigations of pipe flow by R. G. Deissler [7,8,9], who also performed approximate calculations.

3. Mathematical formulation

The theory is only restricted to the treatment of a very thin layer (for example, compared with the boundary layer thickness). Therefore, in the flow equations all average values in planes parallel to the wall (y = const) can be assumed to be constant. Derivatives with respect to x and z are only considered in the fluctuation variables but not in the averages. Let u, v, w be the velocity components in the directions of the x, y and z axis ($\bar{w} \equiv 0$). Let p be the pressure.

The continuity equation is then reduced to

(5) $$\frac{\partial(\overline{\varrho v})}{\partial y} = 0$$

A bar here means time averaging in the following.

The Navier-Stokes equations for the x and y component are written as follows after averaging over time[4]:

(6) $$\frac{\partial(\overline{\varrho u v})}{\partial y} = \frac{\partial}{\partial y}\left[\overline{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}\right],$$

(7) $$\frac{\partial(\overline{\varrho v^2})}{\partial y} = -\frac{\partial \bar{p}}{\partial y} + \frac{\partial}{\partial y}\left[2\overline{\mu\frac{\partial v}{\partial y}} - \frac{2}{3}\overline{\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}\right].$$

In the energy equation we ignore the thermal radiation

(8) $$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial y}\left[\overline{\varrho v(u^2 + v^2 + w^2)}\right] + \frac{\partial}{\partial y}(c_p\overline{\varrho v T}) = \\ &= \frac{\partial}{\partial y}\left(\lambda\frac{\partial \bar{T}}{\partial y}\right) + \frac{\partial}{\partial y}\left[\overline{u\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)} + 2\overline{v\mu\frac{\partial v}{\partial y}} - \right. \\ &\left. - \frac{2}{3}\overline{v\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)} + \overline{w\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)}\right]\end{aligned}$$

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4) The volume viscosity is set equal to 2 μ/3.

In handbooks (for example [10]), usually one finds a different form for the energy equation. In the notation above, we express the fact that for the present special case, that the change in the kinetic energy and the enthalpy equals the energy supply by heat conduction and the work performed by the friction stresses. In addition we have the state equation of the gas. The continuity equation, the Navier-Stokes equations and the energy equation can immediately be integrated over y. Using the additional boundary conditions for y = 0, that is

$$u = v = w = 0, \quad \mu \left(\frac{d\bar{u}}{dy}\right)_w = \tau_w,$$

$$\lambda \left(\frac{d\bar{T}}{dy}\right)_w = q_w, \quad \bar{p} = p_w,$$

we then obtain the following equation system:

Continuity equation:

(9) $$\bar{\varrho} \bar{v} = 0,$$

Navier-Stokes equations:

(10) $$\bar{\varrho} \overline{uv} + \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_w,$$

(11) $$\bar{\varrho} \overline{v^2} + \bar{p} - 2\mu \frac{\partial v}{\partial y} + \frac{2}{3}\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = p_w,$$

Energy equation

(12) $$\begin{aligned} &\frac{1}{2}\bar{\varrho} \overline{v(u^2 + v^2 + w^2)} + c_p \bar{\varrho} \overline{vT} - \lambda \frac{\partial \bar{T}}{\partial y} + \\ &+ u\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2v\mu \frac{\partial v}{\partial y} - \\ &- \frac{2}{3} v\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + w\mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = q_w, \end{aligned}$$

State equation of the gas

(13) $$\bar{p} = \frac{p_\infty}{a_\infty^2 T_\infty} \bar{\varrho} \bar{T}$$

Equation (9) says that the mass flux perpendicular to the wall is zero everywhere. In equation (10) τ_{tu} represents the apparent shear stress due to turbulence motion and $\bar{\mu}(\partial\bar{u}/\partial y + \partial\bar{v}/\partial x)$ is the fraction of the shear stress produced by the viscosity. Accordingly, in (12) $-c_p\rho\overline{vT}$ is the turbulent heat flux and $\bar{\lambda}\partial\bar{T}/\partial y$ is the heat flux produced by the thermal flux. The energy equation (12) can be written in the following manner using (10)

(14) $$-c_p\rho\overline{vT} + \bar{\lambda}\frac{\partial\bar{T}}{\partial y} = q_w - \bar{u}\tau_w - D_t - D_\mu.$$

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Here we have

(15) $$D_t = \frac{1}{2}\overline{\rho v(u^2 + v^2 + w^2)} + \bar{u}\overline{\rho uv}$$

is the kinetic energy flux caused by turbulent motion (turbulent energy diffusion). The expression

(16) $$\begin{aligned} D_\mu = {} & \overline{u\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)} + 2\overline{v\mu\frac{\partial v}{\partial y}} - \\ & - \frac{2}{3}\overline{v\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)} + \\ & + \overline{w\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)} - \bar{u}\overline{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)} \end{aligned}$$

represents the flux of kinetic energy due to viscosity forces (viscous energy diffusion). In the form (14), the energy equation states the following:

The local heat flux $q = -c_p\rho\overline{vT} + \bar{\lambda}\partial\bar{T}/\partial y$ equals the heat flux transferred to the wall q_w, reduced by the work $\bar{u}\tau_w$ of the shear stresses due to turbulent motion and the energy transported by viscosity $D_t + D_\mu$.

Up to this point, we have not introduced any simplifications except for the assumption of the homogeniety with respect to the planes y = const and we ignored the temperature flucutations at the wall.

The apparent shear stress (10), the turbulent heat flux and

the energy diffusion terms in (14) are static averages of fluctuation parameters, where the averaged velocity $\bar{u}$ does not occur. In order to show this, one has to split off the fluctuating parameters in the well known manner, an average value and a fluctuation part ($u = \bar{u} + u'$, $\overline{u'} = 0$ etc.). From Equation (9) we then find

(17) $$\overline{\varrho v} = \bar{\varrho}\bar{v} + \overline{\varrho' v'} = 0.$$

Accordingly, one can also express the average $\bar{v}$ by the correlation between the density and the velocity fluctuations. In this way the apparent sheer stress becomes

(18) $$-\overline{\varrho u v} = -\bar{\varrho}\,\overline{u'v'} = -\frac{\overline{\varrho' v'}\;\overline{\varrho' u'}}{\bar{\varrho}} - \overline{\varrho' u' v'}.$$

For the turbulent heat flux in the same way we obtain

(19) $$-\overline{\varrho v T} = -\bar{\varrho}\,\overline{v'T'} = -\frac{\overline{\varrho' v'}\;\overline{\varrho' T'}}{\bar{\varrho}} - \overline{\varrho' v' T'}.$$

The turbulent energy diffusion is written as

(20) $$D_t = -\frac{1}{2}\bar{\varrho}\,\overline{v'(u'^2 + v'^2 + w'^2)} = \overline{\varrho'(\ldots)}.$$

and the viscous energy diffusion finally becomes

(21) $$\begin{aligned} D_\mu = {} & \frac{\bar{\mu}}{2}\frac{d\,\overline{(u'^2 + v'^2 + w'^2)}}{dy} + \bar{\mu}\frac{d\overline{v'^2}}{dy} + \\ & + \frac{5}{3}\overline{\mu v'^2}\,\frac{d\bar{\varrho}/dy}{\bar{\varrho}} + \overline{(\mu', \varrho', \ldots)}. \end{aligned}$$

The terms contained in the density fluctuations are not written out in (20), because they are very cumbersome and will not be used in the following. In addition, in (21), $\overline{(\mu', \varrho', \ldots)}$ is only shown for simplicity. This means that additional terms occur which are related to the fluctuations of the material values. In the similar manner we can write down the sheer stress caused by viscosity and the heat flux produced by heat conductivity

(22) $$\begin{aligned} & \overline{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)} = \bar{\mu}\frac{d\bar{u}}{dy} + \overline{\mu'\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)}, \\ & \overline{\lambda\frac{\partial T}{\partial y}} = \bar{\lambda}\frac{d\bar{T}}{dy} + \overline{\lambda'\frac{\partial T'}{\partial y}}. \end{aligned}$$

4. Calculations

The equation system (10), (11), (13) and (14) can only be treated further with special assumptions about the turbulence mechanism. In order to provide a basis of discussion and to qualitatively the influences, we perform calculations with the following assumptions:

1. The fluctuations in the material parameters ρ', μ', λ' can be ignored.

2. The gradient of the average velocity follows the relationship

$$\frac{d\bar{u}}{dy} = \frac{\tau_w}{\bar{\mu}} f'(\eta), \tag{23}$$

where $f'(\eta)=df/d\eta$ is the same dimensionless function as for incompressible flow, equation (1). There is a difference that the argument η depends on the local changing material parameters.

3. The ratio σ_t of the coefficients for turbulent momentum and heat exchange (turbulent Prandtl number) is constant.

4. The diffusion terms $D_t + D_\mu$ can be ignored.

5. The fluctuation and viscosity terms in Equation (11) can be ignored.

The influence of the density fluctuations will be discussed later on. Equation (23) is based on the assumption that the mechanism of turbulence can be controlled by the local conditions; that is, that the quantity $-\overline{\rho u' v'}$ is only a function of

$$\eta = \frac{\rho}{\bar{\mu}} \sqrt{\tau_w \bar{\rho}} \int_0^y \frac{dy}{\bar{\rho}} \tag{24}$$

and does not depend explicitly on M_τ and β_q. The effect of

compressibility and temperature on the velocity distribution therefore is only implicit through the changing material quantity which vary with η. If we ignore the viscosity fluctuations, Equation (10) is reduced to

$$(25) \qquad -\overline{\varrho u v} + \bar{\mu}\frac{d\bar{u}}{dy} = \tau_w .$$

The comparison with (23) gives f' the following meaning

$$(26) \qquad f'(\eta) = 1 + \frac{\overline{\varrho u v}}{\tau_w} ,$$

since the turbulent velocity fluctuations at the wall vanish, here we have f'(0) = 1. For large η, f'(η) asymptotically becomes the following

$$(27) \qquad \lim_{\eta \to \infty} f'(\eta) = \frac{1}{\varkappa \eta} .$$

Data has been published about the variation of f'(η) over the entire η range based on measurements by various authors. For example, H. Reichardt [11], D. Coles [12], E. R. van Driest [13]. /268
For the numerical calculations we use the relationship given by van Driest [13]

$$(28) \qquad f' = \frac{2}{1 + \sqrt{1 + 4\varkappa^2 \eta^2 [1 - \exp(-\eta/A)]^2}}$$

and the universal constants x = 0.4 and A = 26.

The third assumption is a hypothesis used often. Since according to the exchange law of Boussinesq or Schmidt, we best write this assumption in the following form

$$(29) \qquad \frac{\overline{\varrho u v}}{\overline{\varrho v T}} = \sigma_t \frac{d\bar{u}/dy}{d\bar{T}/dy} = \sigma_t \frac{d\bar{u}}{d\bar{T}} .$$

According to the measurements of H. Ludwieg [14], it seems that $\sigma_t = A_t/A_\eta = 0{,}9$ is a suitable value. We find the following from (26) and from (29) for the turbulent heat flux

$$(30) \qquad -\overline{\varrho v T} = -\frac{\overline{\varrho u v}}{\sigma_t}\frac{1}{d\bar{u}}\frac{d\bar{T}}{} = \tau_w \frac{1 - f'(\eta)}{\sigma_t}\frac{d\bar{T}}{d\bar{u}} .$$

We will discuss the influence of $D_t + D_\mu$ later on. By ignoring these terms, the energy equation (14), using equation (30) and

(23) gives the following relationship between the temperature and the velocity distribution

(31) $$\left\{\left[1-f'(\eta)\right]\frac{1}{\sigma_t}+f'(\eta)\frac{1}{Pr}\right\}c_p T_w \frac{d\bar{T}}{d\bar{u}} = q_w - \bar{u}\tau_w.$$

The ignored terms introduced in 5. apparently are not important. The pressure is assumed to be constant; accordingly: $\bar{p} = p_w$. If we at the same time ignore the density fluctuations, the equation of state (13) of the gas is reduced to the generally used form for boundary layers

(32) $$\bar{\varrho}\,\bar{T} = \varrho_w T_w,$$

which then specifies the density as a function of temperature.

Equations (23), (24), (28), (31), (32) and (3) now represent a solvable system. Mathematically, we are dealing with the simultaneous integration of two ordinary differential equations of the first order with given initial conditions. Numerical solutions for this equation system have been found at the Goettinger electronic computer G1[5]. In addition to $\sigma_t = 0.9$, we also have $\gamma = 1.4$, $Pr = 0.72$, and $\omega = 0.76$.

For the completely turbulent range, where no turbulent forces are active except in turbulent dissipation, the equation system can be solved in closed form. We have the following asymptotic solution

(33) $$\bar{T} = T_w\left[\sigma_t \beta_q \bar{u}_j/u_\tau - \sigma_t \frac{\gamma-1}{2} M_\tau^2\left(\frac{\bar{u}}{u_\tau}\right)^2 + C_1\right].$$

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5) The computer G1 was made available by the Max Planck Institute for Physics, Astrophysics Dept. for these calculations. We would like to thank Prof. L. Biermann at this point.

(34)

$$\bar{u} = u_\tau \left[\frac{1 + C_1}{\sqrt{\sigma_1 \frac{\gamma-1}{2} M_\tau^2}} \times \sin\left[\sqrt{\sigma_1 \frac{\gamma-1}{2}} M_\tau \left(\frac{1}{\varkappa} \ln \eta_w + C_2 \right) \right] + \frac{\beta_q}{(\gamma-1) M_\tau^3} \times \left\{ 1 - \cos\left[\sqrt{\sigma_1 \frac{\gamma-1}{2}} M_\tau \left(\frac{1}{\varkappa} \ln \eta_w + C_2 \right) \right] \right\} \right].$$

Here C_1 and C_2 are still unknown integration constants. For the case $M_\tau = 0$ we find

(35)

$$\bar{u} = u_\tau \left\{ 1 + C_1 \left(\frac{1}{\varkappa} \ln \eta_w + C_2 \right) - \frac{\sigma_1 \beta_q}{4} \left(\frac{1}{\varkappa} \ln \eta_w + C_2 \right)^2 \right\}.$$

This equation for $\beta_q = 0$ again is reduced to the well known relationship

(36)

$$\bar{u} = u_\tau \left[1 + C_1 \left(\frac{1}{\varkappa} \ln \eta_w + C_2 \right) \right].$$

For incompressible flow, the values in equation (28) are $C_1 = 1$, $C_2 = 5.2$. In general, C_1 and C_2 are functions of M_τ and β_q.

Figures 1 to 4 show a few calculated results. Figures 1a and 1b show the velocity distribution $\bar{u}/U_\tau$ as a function of η_w in the usual semi-logarithmic plot and the corresponding temperature distribution $\bar{T}/T_w$ for different values of the heat transfer coefficient β_q for zero Mach number, $M_\tau = 0$. From this it follows that the velocity for constant wall distance with β_q increases. In other words, when heat is transferred from the flow to the wall. The temperature of course decreases when one approaches the wall, if heat is transferred to the wall. Figures 2a and 2b show the influence of the Mach number M_τ for zero heat transfer coefficient $\beta_q = 0$. Here $\bar{u}/u_\tau$ becomes smaller for a given η_w if M_τ increases. The temperature distribution at the wall has a horizontal tangent and decreases with η_w. The reason for this is that the heat produced by friction, which is large for large velocities, has to be led off into the flow to the outside. The two parameters accordingly have opposing effects on

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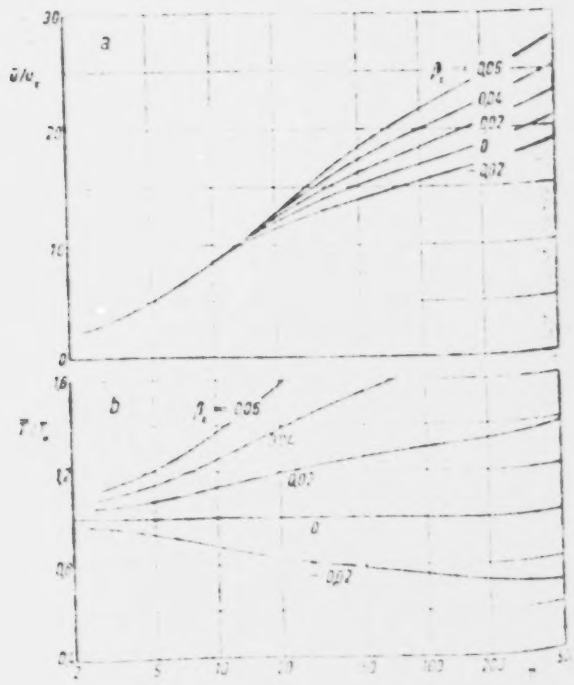


Figure 1. Theoretical velocity distribution (a) and temperature distribution (b). Influence of thermal heat transfer coefficient β_q for zero Mach number, $M_\tau = 0$.

the flow. Figures 3a and 3b show the combined influence of M_τ and β_q. The temperature distribution for positive β_q has a maximum. In some cases we also show the asymptotic solutions according to (33) and (34) in these figures. The quantities C_1 and C_2 which are required so that (33) and (34) do actually agree with the asymptotes of the numerical solutions, are shown in Figures 4a and 4b as a function of β_q and M_τ. Here again we show the opposing effect of β_q and M_τ.

5. Temperature distribution according to experiments

We will now compare the calculations with the available results of hypersonic tunnel measurements [4], first of all, the temperature distribution. After conversion of the similarity law (4) the local stagnation temperature can be represented in a dimensionless form as a function of the local Mach number $M = \bar{u}/\bar{a}$ where is the time average of the local speed of sound:

$$\frac{\bar{T}_s}{T_\infty} = \Theta\left(\frac{\bar{M}}{M_\tau}, M_\tau, \beta_w, Pr, \gamma, \omega\right). \tag{37}$$

We selected this method of representation because the two quantities $\bar{T}_s$ and $\bar{M}$ are determined independently in experiments. In the plot (Figure 5a) the numerical solutions show systematic deviations from the measurements which is expressed especially in the region outside of the lower layer ($\bar{M}/\bar{M}_\tau > 15$), primarily in the form of a parallel displacement. For the other measurement /270

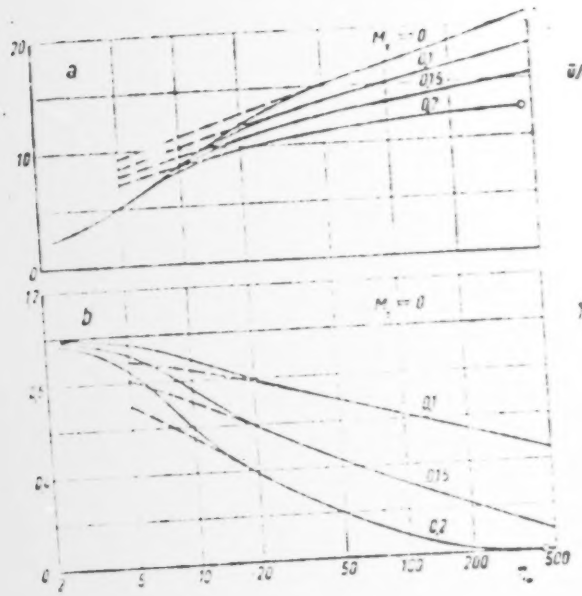


Figure 2. Theoretical velocity distribution (a) and temperature distribution (b). Influence of Mach no. M_τ and zero heat transfer coefficient $\beta_q=0$.

—numerical integration,
--asymptotic solution,
Eqs. (33), (34)

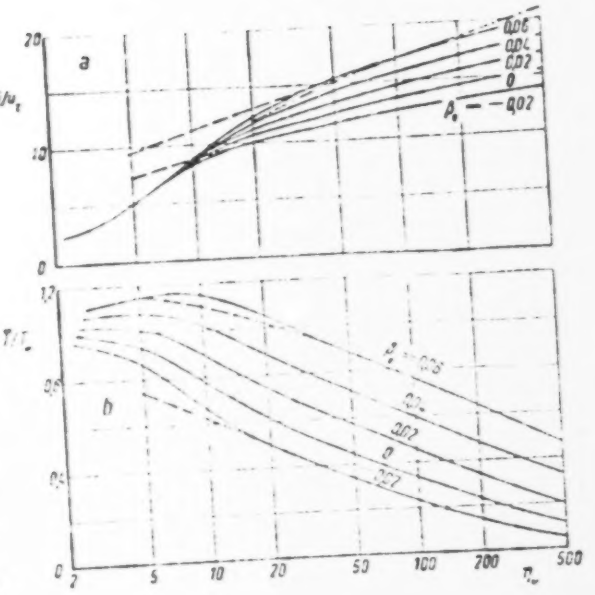


Figure 3. Theoretical velocity distribution (a) and temperature distribution (b). Combined influence of heat transfer coefficient β_q and Mach no. M_τ.

—numerical integration, M =0.15
--asymptotic solution, Eqs (33),(34)
-.-incompressible flow $\beta_q=0, M_\tau=0$

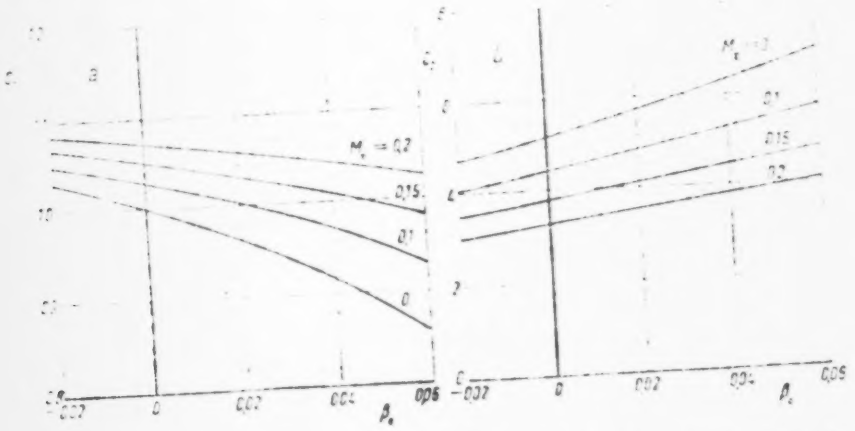


Figure 4. Theoretical magnitude of the integration constants C_1 for the asymptotic temperature law (33) and C_2 for the asymptotic velocity law (34) as a function of the heat transfer coefficient β_q and the Mach no. M_τ

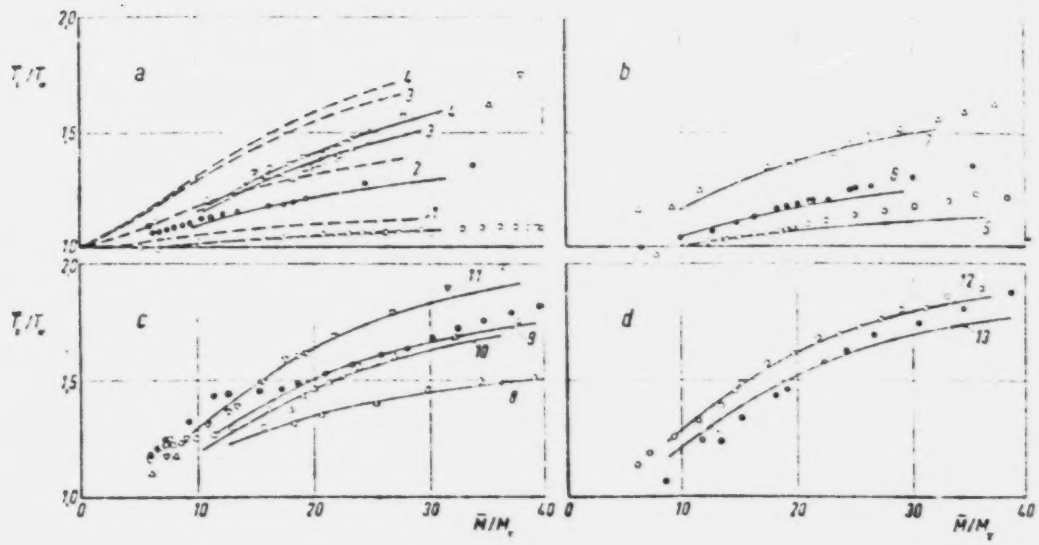
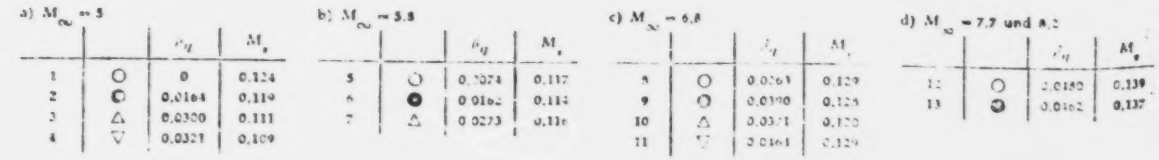


Figure 5. Distribution of the local stagnation temperature $\bar{T}_s/T_w$ as a function of local Mach number $\bar{M}/M_\tau$. Measurement points according to R. K. Lobb, E.M. Winkler, J. Persh [4].
——asymptotic temperature law (33) with suitable selected C_1
---theory according to Section 4 (numerical integration)



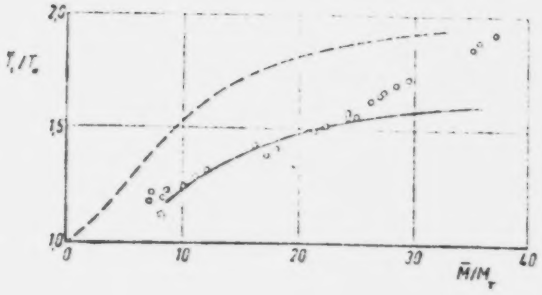


Figure 6. Distribution of the local stagnation temperature $\bar{T}_s/T_w$ as a function of local Mach no. $\bar{M}/M_\tau$. Measurement points according to F. K. Hill [15]
——asymptotic temperature law (33) with suitably selected C_1
---theory according to Sec. 4 (numerical integration)

series of [4], we obtain almost the same picture. Therefore, $\bar{T}_s$ was plotted according to the asymptotic solution (33) with suitable selection of C_1, so that in the region in which we can assume that the assumptions for completely turbulent wall flow are satisfied, we have the best possible agreement with the measured values (Figures 5a to 5d). Unfortunately, this region is not very large as already mentioned. This is because the viscous sublayer in these

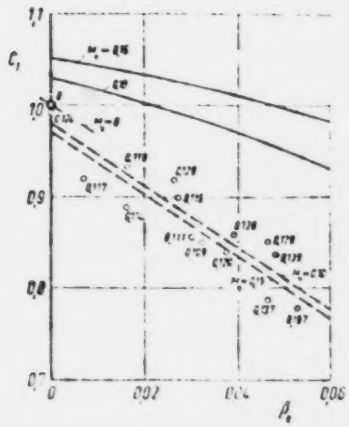


Figure 7. Integration constant C_1 for the asymptotic temperature law (33) according to test evaluations and theory
——numerical integration
---equalization line according to (38)
numbers at the test points M_τ

measurements was relatively thick, so that the assumptions for the wall law are not very well satisfied. Nevertheless, in almost all cases we have a finite region in which the calculations follow the measured points. We do not have such complete information about the measurements of F. K. Hill [15]. The evaluation of a measurement series is shown in Figure 6. The heat transfer density q_w had to be estimated from the wall temperature (see Appendix). Here again we have a region where the variation of the asymptotic solution (33) agrees with the measured results, even though due to the reduced Reynolds number the range of validity is even smaller for the wall law than for the NOL measurement [4].

The C_1 values which are required so that the asymptotic temperature law (33) matches the measurements is shown in Figure 7 as a function of β_q. They show a dependence on β_q in the same sense as the values according to the numerical calculations, which have a larger magnitude. The influence of M_τ cannot be so clearly determined from the measurements, because this parameter was varied only slightly during measurements and the measurements show scatter. Assuming a linear dependence of the C_1 values on β_q and M_τ and with consideration of the condition $C_1 = 1$ for the incompressible case, if we apply the method of least squares

(38) $C_1 = 1 - 3{,}4\beta_q - 0{,}2M_\tau$.

The average error is ± 0.024. Differences between the theoretical and the experimental C_1 values can hardly be explained with the value of σ_t which could vary in the sublayer. σ_t will then have to take on very large values. We do not expect a great effect from the density fluctuations which were not considered because the corresponding terms are very similar for the apparent sheer

stress, Equation (18), and the turbulent thermal flux, Equation (19). Also the fact that the exact expression

$$c_p \mu \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \Big/ \lambda \frac{dT}{dy}$$

was replaced by the simplified form $Pr\,(d\bar{u}/dy)/(d\bar{T}/dy)$ (see (22), can hardly be held responsible for the discrepancies. Therefore, it is natural the reasons for the differences are suspected to be the diffusion terms D_t and D_μ which were ignored in the calculations. These would have to have a positive sign according to energy transport in the direction towards the wall, as would be expected in the sublayer as well. A rough estimation of this influence can be performed if one sets the energy transported by turbulent motion proportional to the gradient of the fluctuation energy according to L. Prandtl [16], that is

$$D_\mu + D_t \quad (\bar{\mu} + \varepsilon_e) \frac{d\,\frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2}}{dy}$$

where ε_e is the exchange variable for energy, which has about the same magnitude as the change variable for momentum [6] or the heat. From the energy equation, we find the following temperature change ΔT caused by the diffusion terms

$$(c_p \varepsilon_\theta + \bar{\lambda}) \frac{d\,\Delta T}{dy} = -(\bar{\mu} + \varepsilon_e) \frac{d\,\frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2}}{dy} \tag{39}$$

If one estimates the quotient $(\bar{\mu} + \varepsilon_e)/(c_p \varepsilon_\theta + \bar{\lambda}/c_p)$ to be constant and equal to one, then equation (39) can be integrated:

$$c_p \Delta T = -\frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2} \tag{40}$$

The parallel displacement of the temperature curve in the completely turbulent range, that is the change of C_1, can then be

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6) According to an evaluation of Wieghardt [16], ε_e for energy exchange equals 0.7 times the exchange parameter for momentum.

expressed by

(41) $$\Delta C_1 = \frac{\Delta T}{T_w} = -(\gamma - 1) M_\tau^2 \frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2 u_\tau^2}$$

According to measurements of J. Laufer [17] for incompressible flow and in the completely turbulent range, we have the wall law

$$\frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2 u_\tau^2} = 4.5$$

With this value, we obtain $\Delta C_1 = -0.02$ to -0.06 for the measurements investigated. Observed fluctuations between measurement and theory are larger and differ by the ΔC_1 estimated in this manner by factors between 2 to 5. It is thinkable that the energy transport by diffusion is actually greater than given by this estimation. However, there are probably other influences as well. It is remarkable that in (41) we have the factor M_τ^2. Through the diffusion terms we now feel the opposite trend of M_τ in the calculations. Therefore, we can explain that the growth of C_1 with M_τ (Figure 4a) determined in calculations was not found in measurement evaluations. The method of least squares according to (38) even shows a decrease of C_1 with M_τ. In any case, from these comparisons one can conclude that the energy transport in the sublayer described by D_t and D_μ apparently makes a contribution to the energy equilibrium which cannot be ignored. In the completely turbulent part, the conditions are well represented by the simplified calculation assumptions.

6. Velocity distributions according to experiments

The measured velocity distributions according to form (4) are shown in Figures 8a to 8d and Figure 9 using the semi-logarithmic representation. The velocity variation according to the numerical integrations is also shown in these figures. Accordingly, the influence of the changing material parameters is shown in a qualitatively correct form in the calculations, but there are quantitative differences. The error in the temperature distribution of course also has an effect on the velocity distribution,

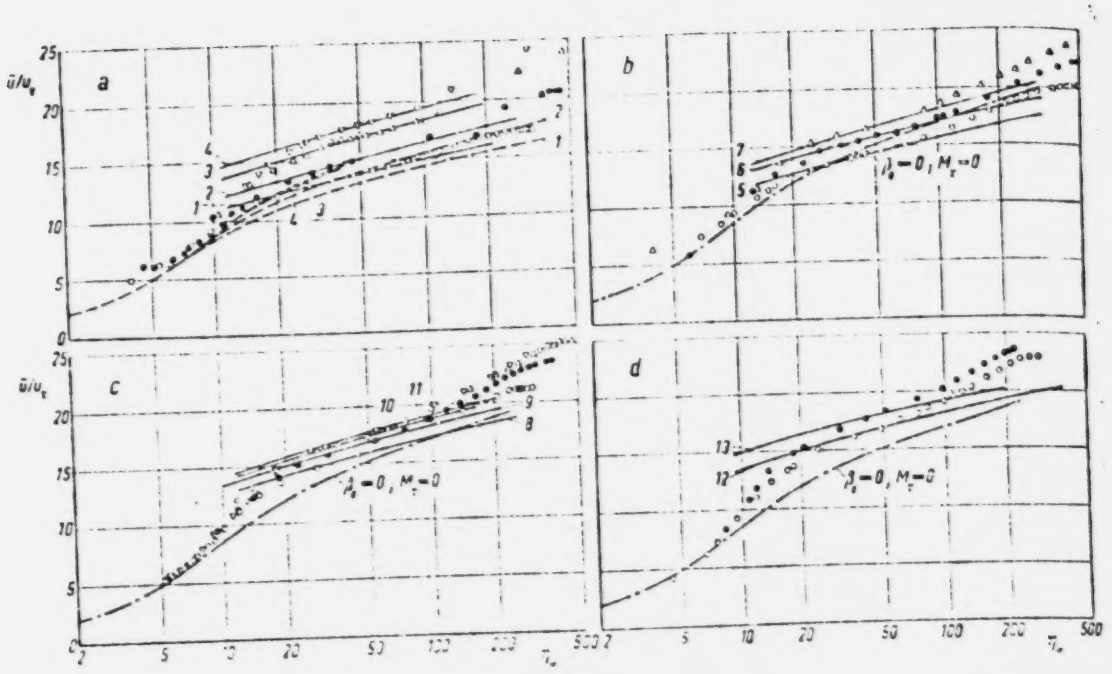

Figure 8. Velocity distribution $\bar{u}/u_\tau$ as a function of the wall distance $\eta_\infty = \frac{y}{\nu}\sqrt{\frac{\tau_w}{\rho}}$. Measurement points according to R. K. Lobb, E. M. Winkler, J. Persh [4].
——asymptotic velocity law (34) with suitably selected C_2 and C_1 as in Fig. 5.
---theory according to Sec. 4 (numerical integration)
-·-incompressible flow $\beta_q=0$, $M_\tau=0$

a) $M_\infty = 5$

		β_q	M_τ
1	○	0	0.124
2	●	0.0164	0.119
3	△	0.0300	0.111
4	▽	0.0321	0.109

b) $M_\infty = 5.9$

		β_q	M_τ
5	○	0.0274	0.117
6	●	0.0162	0.111
7	△	0.0273	0.116

c) $M_\infty = 6.5$

		β_q	M_τ
8	○	0.0263	0.125
9	●	0.0390	0.120
10	△	0.0371	0.129
11	▽	0.0464	0.129

d) $M_\infty = 7.7$ und 8.2

		β_q	M_τ
12	○	0.0480	0.139
13	●	0.0462	0.137

but it does not explain the deviations between the theory and the experiment. On the other hand,the asymptotic solution (34) also here matches the measurements well for the completely turbulent part in most cases, if for C_1 one uses the values found for the previous evaluations and the constant C_2 are adjusted accordingly. For incompressible boundary layers, the range of validity of the wall law seems to extent over 1/5 of the boundary layer total thickness. In the measurements of Hill [15] the lower layer influence by the viscosity already takes up at least 1/3 of the total thickness. The variation of the velocity distribution in

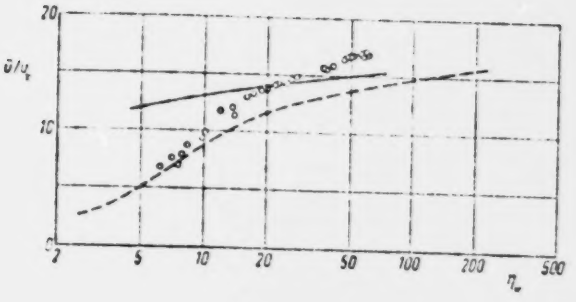

Figure 9. Velocity distribution u/u_τ as a function of the wall distance 22
Measurement points according to F. K. Hill [15]
——asymptotic velocity law (34) with suitably selected C_2 and C_1 as in Fig. 6.
---theory according to Sec. 4 (numerical integration)
M_∞=9.07, β_q=0.053, M_τ=0.187

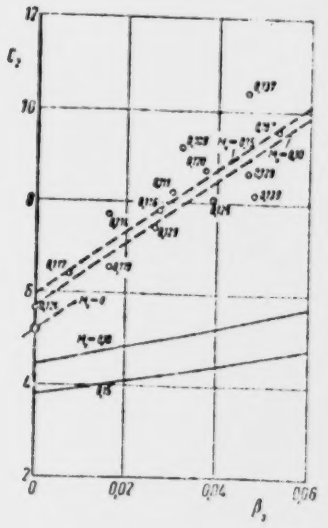

Figure 10. Integration constant C_2 for the asymptotic law (34) after evaluation of experiments and theory.
——numerical integration
---equalization line according to (42) numbers near the test points: M_τ

the completely turbulent part of the wall law can therefore also be estimated in this case (Figure 9). /272

The C_2 values corresponding to these comparisons (Figure 10) depend very greatly on β_q. Figure 10 shows the calculated values according to the numerical integrations. Just like the integration constants C_1, the C_2 values determined with the two methods show the same trend regarding the dependence on β_q. They however show quantitative differences. If we again assume a linear dependence on β_q and M_τ and if we determine that in the incompressible case we have $C_2 = 5.2$, then the method of the least squares gives

(42)
$$C_2 = 5{,}2 + 70\beta_q - 5M_\tau.$$

The average error is ±0.7. The influence of M_τ in the measurements is smaller than for the numerical calculations and has the opposite sign. It is remarkable that the C_2 value of the measurements of Hill corresponds completely

to the other experimental values even though the velocity distribution of this measurement deviates much less from the incompressible law than the distributions of the NOL measurements in spite of the larger Mach number.

Just like for the temperature distributions, the assumptions of the calculation are least satisfied in the lower layer, where the viscosity and the turbulence participate about equally to the sheer stress. Considering the information about the turbulence, it is not possible to determine the reasons with certainty. It is natural to assume that the fluctuations of the material parameters, especially the density fluctuations, which was pointed out by T. Y. Li and H. T. Nagamatsu [18], are influencing factors in an important way. Therefore, we will make a few remarks about this.

According to the state equation of the gases, we have the following relationship between the pressure, density and temperature fluctuations

$$p'/\bar{p} \approx \varrho'/\bar{\varrho} + T'/\bar{T}.$$

The velocity fluctuations are very small just like for incompressible flows. This means that the pressure distributions also remain small and the relative density fluctuations will be of the same order of magnitude as the relative temperature fluctuations and will be statistically correlated with them. This fact is also justified by the assumption that the entropy fluctuations correlate stronger with the velocity fluctuations than with the primarily isentropic pressure fluctuations. If one does not have extremely high Mach numbers and heat transfer rates, the temperature fluctuations and therefore also the density fluctuations will be relatively small $|T'| \ll \bar{T}$. Under the assumptions, the errors which are created by ignoring the density fluctuations remain small. Nevertheless, it is possible that they will not be small compared with changes which the Mach number and the heat transfer cause compared with incompressible flow. In order for

this to be the case we have to have $T' \ll |\bar{T} - T_w|$. It is questionable whether for the smaller wall distances, that is in the sublayer, this is the case, because here $|\bar{T} - T_w|$ is only small. For large wall distances, $|\bar{T} - T_w|$ has the order of magnitude of $\bar{T}$. Therefore, the density fluctuations in the sublayer can become effective. In a similar way, the viscosity contributions, which are also proportional to the temperature fluctuations, could become important. This is one possible explanation of why the simplified calculations give a very good representation of the conditions in the completely turbulent region, but there are deviations in the sublayer.

It is also thinkable that the variation of the averages of the material quantities have a direct influence on the turbulence mechanism. The application of the stationary stability theory to the laminar boundary layer for compressible flow [19], for example, gives an increase in the stability, if the heat transfer from the fluid to the wall takes place. Heat transfer from the wall to the liquid reduces stability on the other hand. Perhaps in turbulent flows, there are similar effects in the sublayer, so that for positive β_q the turbulent velocity fluctuations decay at a larger distance from the wall (thickening of the sublayer) then for $\beta_q = 0$ or for negative β_q. Such a behavior would be another very plausible explanation for the observed deviations between the calculation and the experiment.

In these discussions one has to consider that there are certain problems associated with the test result. Already in the incompressible case, there is no unity among the various researchers about the size of the constants $\varkappa$ and C_2. For example, in the references there are values of C_2 which vary between 4.1 and 6.5. The experiments used in the present analysis certainly have been performed with great care. In spite of this, there could be certain quantitative contradictions in the measurements in Reference [4] and also compared with the results from other sources, which cannot be overlooked competely. First of all, we notice

several local irregularities in the temperature variation close to the wall, which cannot be brought into agreement with the theoretically expected temperature distribution, and varies widely among the individual measurement series. Therefore, we believe that the temperature measurements very close to the wall are not as reliable. The very substantial scatter of the measurement series on the large scale as regards the influence of β_q and M_τ are much more important than the local irregularities. For example this can be seen in the scatter of the constants C_1 and C_2 according to Figures 7 and 10. We therefore fear that unknown perturbation influences are contained in these experiments. Finally, we should also recall the fact that the local friction coefficient for the same Mach number according to the theory [6] and according to experiments [20] increases when the wall is cooled. This trend cannot at all be found from the measurements [4]. Considering this quantitative deficiency of the test results, it seems that it is not possible to clarify the discrepancies between tests and calculations without procuring additional experimental material.

In the application of the wall law to boundary layer problems, the velocity and temperature distribution in the sublayer are not very interesting in general. Instead, usually the distributions in the completely turbulent region are required in order to be able to fit together the outer boundary layer parts and the parts near the wall (for example see [21]). In this region, the asymptotic solutions according to equations (33), (34) and (35) give a very good prediction. It is recommended to assume that the constants C_1 and C_2 are given by (38) and (42) and to set

$$\sigma_t = 0.9, \varkappa = 0.4.$$

7. Summary

The known similarity law for the turbulent flow at the wall was expanded for compressible flow media as regards the influence of Mach number and heat transfer. As additional parameters, we

have introduced the Mach number M_τ formed with the sheer stress velocity and a dimensionless heat transfer coefficient β_q. /274 With simplifying assumptions, calculations were performed and we showed the influence of the parameters. In the comparison of calculations with published measurements it was found that the calculation assumptions in the competely turbulent region described the actual conditions with sufficient accuracy. In the viscous sublayer there are deviations in the temperature distributions which can be primarily attributed to the energy diffusion which is ignored in the calculation. The differences in the velocity distributions apparently are caused by the turbulent fluctuations of density which was not considered in the calculations and also possibly by the viscosity and a turbulence mechanism which cannot be controlled. The velocity and temperature distributions in the completely turbulent region can be calculated according to the formulas given.

8. Appendix. Data from the test series

In order to identify the test series used in the original references [4] and [15], the following table gives the most important data.

Measurement Series	M_∞	$\frac{T_e - T_{ic}}{T_c}$	M_t	β_q
1	4,93	0	0,124	0
2	5,01	0,224	0,119	0,0164
3	5,03	0,374	0,111	0,0300
4	5,06	0,420	0,109	0,0321
5	5,75	0,108	0,117	0,0074
6	5,79	0,238	0,114	0,0162
7	5,82	0,379	0,116	0,0273
8	6,83	0,326	0,129	0,0263
9	6,78	0,438	0,128	0,0390
10	6,83	0,444	0,120	0,0371
11	6,78	0,499	0,129	0,0461
12	7,67	0,488	0,139	0,0480
13	8,18	0,495	0,137	0,0462
14	9,07	0,438 *)	0,187	0,0530

*) $(T_e - T_{ic})/T_\infty = 0,43$.

M_∞ = Mach number of the flow outside of the boundary layer.

T_e = Equilibrium wall temperature (for $q_w = 0$).

The measurement series 1 to 13 were taken from [4], and measurement series 14 was taken from [15].

In the test series 4, 12 and 14, the heat flux density q_w was estimated as follows

$$St_\infty / c_f = 0,57.$$

Stanton number:

$$St_\infty = \frac{q_w}{(T_e - T_w)\, c_p\, \varrho_\infty\, u_\infty},$$

local friction coefficient

$$c_f = \frac{2\tau_w}{\varrho_\infty\, \bar{u}_\infty^2}.$$

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